The Definition of a Function of a Real Number

Let A and B be sets. If f is a relation from A to B then f is called a *function* from A to B if for every $a \in A$ there exists a unique $b \in B$ such that $(a, b) \in f$.

Or equivalently,

If A and B are sets, then a *function*, f, from A to B is the set of ordered pairs (a, b) contained in $A \times B$ such that if $(a, b) \in f$ and $(a, b') \in f$ then $b = b'$.

Please take note of the following:

- The notation $f : A \to B$ is used to demonstrate that f *maps* A into B.
- Set A is called the *domain* of the function and usually notated $D(f)$.
- The image of a under f is notated $f(a) = b$ and all elements of this set is called the range and can be notated $R(f) = \{f(x) \mid x \in D(f)\}$.
- The range is a subset of B.

Questions:

1. The second formulation of the definition of a function of a real number makes use of the Cartesian or set product. How is this product defined?
2. What subset of the Cartesian plane is defined under $A \times B$ if:
   (a) $A = \{1, 2, 3, 4\}$ and $B = \{0, 1, 2, 3, 4\}$
   (b) $A = \{1, 2, 3\}$ and $B = \mathbb{R}$
   (c) $A = [-2, 3]$ and $B = [5, 8]$
   (d) $A = \mathbb{R}$ and $B = \{-3, -1, 1, 3, 5\}$
3. Let f be the squaring function such that $f : A \to \mathbb{R}$. What is the range or the image of A under f if $A =$
   (a) $\{-5, -3, -1, 0, 2, 4, 6, 8\}$
   (b) $\mathbb{R}$
   (c) $\{2, 3\} \cup [6, 11]$
4. When mapping the function f in the Cartesian plane does the graph "appear" any different than $A \times \mathbb{R}$?
5. Which of the domains, if any, in question 3 result in an onto function? Which of the domains, if any, result in a function that is one-to-one?
6. Can a function be onto and not one-to-one or be one-to-one and not onto? What is the significance of these properties?

Definitions Frequently Used in Association with Functions

If $a, b \in \mathbb{R}$ and $a < b$ then $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$ is called an *open interval*, and a and b are called *endpoints*.

If $a, b \in \mathbb{R}$ and $a < b$ then $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ is called a *closed interval*, and a and b are called *endpoints*.

A function $f : A \to B$ is called *onto* or *surjective* if and only if for every $y \in B$ there exists an $x \in A$ such that $y = f(x)$.

A function $f : A \to B$ is called *one-to-one* or *injective* if and only if for every $x_1, x_2 \in A$ if $f(x_1) = f(x_2)$ then $x_1 = x_2$

Geometrically speaking, a function $f : A \to B$ is surjective if it passes the first horizontal line test, namely, if $b \in B$ then the horizontal line $y = b$ intersects f at least once. f is injective if it passes the second horizontal line test, namely, the horizontal line $y = b$ intersects f at most once.